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Portfolio Selection Theory with Different Interest Rates for Borrowing and Lending

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Abstract. This paper considers the portfolio selection problem with different interest rates for borrowing and lending. The portfolio frontier is described under the general condition that the riskless borrowing rate is higher than the riskless lending rate.

Key words: different interest rates for borrowing and lending, Kuhn-Tucker condition, portfolio selection, quadratic program

1. Introduction

The theory of portfolio selection of Markowitz (1952) applied mean and variance to characterize return and risk for a combination of more than two financial assets traded in a frictionless economy. Unlimited short selling is allowed and the rates of return on these assets are assumed to have finite variances. Rothschild and Stiglitz (1970, 1971) introduced the concept of second degree stochastic dominance. They showed that when there are more than two risky assets, if there exists a portfolio of assets that second degree stochastically dominates all the portfolio with the same expected rate of return, then this dominant portfolio must have the minimum variance among all the portfolios. This observation is one of the motivations for characterizing portfolios that have the minimum variance for various levels of expected rate of return. The study of mean-variance efficiency by Gonzalez-Gaverra (1973), Merton (1972) and Roll (1977) expanded Markowitz's model to discuss various issues in portfolio management. To understand the formulation of mean-variance as return-risk, Chamberlain (1983) made an effort to characterize the complete family of probability distributions that are necessary and sufficient for the expected utility of terminal wealth to be a function only of the mean and variance of terminal wealth or for mean-variance utility functions. Epstein (1985) shows that mean-variance utility functions are implied by a set of decreasing absolute risk aversion postulates.

The original result of Markowitz was derived in a discrete time, frictionless economy with the same interest rates for borrowing and lending. In reality, investors may be charged a higher interest rate for borrowing money than the interest rate for saving money. Even though many research works assume the same riskless interest rate for borrowing / lending, the discrepancy between borrowing and lending is crucial for the operations of financial institutions. This paper extends the portfolio study of Markowitz to the case of different interest rates for borrowing and lending. In Section 2 we introduce notations and definitions under the context of portfolio theory of Markowitz. In Section 3 we examine the problem with different interest rates for borrowing and lending where the riskless borrowing rate is higher than the riskless lending rate. The programming problem for establishing the portfolio frontier is solved in the closed form by the Kuhn–Tucker condition. Section 4 concludes the paper with remarks and discussions.

This paper considers a nonsmooth two-extrema problem in Section 3. The problem has a simple structure and can be solved quite easily using quadratic programming by solving two quadratic problems. The soluations of the two problems work out our main programming problem.

2. Preliminaries

We follow the notations in Huang and Litzenberger (1988). In general, we consider $N+1$ assets: N risky assets and 1 riskless asset with different interest rates for borrowing and lending. Unlimited short selling is allowed and that the rates of return on these assets have unequal expectation and finite variances. In this section, we first review the case without the riskless asset to introduce some notations and properties useful to our discussion; then, we review the case of the same interest rate for the riskelss asset. Notations and results established here will be useful in the next section.

2.1. PORTOFOLIO FRONTIER WITHOUT RISKLESS ASSET

The random rate of return on the n-th risky asset is $\tilde{r}_n$ for any $n \in \mathcal{N} = \{1, \cdots, N\}$. Its expected rate and variance are $E[\tilde{r}_n]$ and $\sigma^2(\tilde{r}_n)$, respectively. Let $\tilde{r}$ denote the N-vector of rates of return on the N risky assets, e denote the N-vector of expected rates of return on the N risky assets and V the variance-covariance matrix:

$$\tilde{r} = \begin{pmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_N \end{pmatrix} \qquad e = \begin{pmatrix} E[\tilde{r}_1] \\ \vdots \\ E[\tilde{r}_N] \end{pmatrix} \qquad V = \begin{pmatrix} cov(\tilde{r}_1, \tilde{r}_1) & \cdots & cov(\tilde{r}_1, \tilde{r}_N) \\ \vdots & & \vdots \\ cov(\tilde{r}_N, \tilde{r}_1) & \cdots & cov(\tilde{r}_N, \tilde{r}_N) \end{pmatrix}.$$

That is, $e = E[\tilde{r}]$ and $V = cov(\tilde{r}, \tilde{r})$. It is also assumed that the rate of return on any asset as a random variable cannot be expressed as a linear combination of the rates of return on other assets (as random variables). Under this assumption, asset returns are said to be linearly independent and their variance-covariance matrix V is nonsingular. The matrix V is thus positive definite (in general it is positive semidefinite). The rate of return on portfolio p is $\tilde{r}_p = \sum_{n\in\mathcal{N}} w_n \tilde{r}_n = w_p^\top \tilde{r}$. The expected rate of return on portfolio p is $E[\tilde{r}_p] = \sum_{n\in\mathcal{N}} w_n E[\tilde{r}_n] = w_p^\top e$ and its variance is $\sigma^2(\tilde{r}_p) = \sigma^2(w_p^\top \tilde{r}) = cov(w_p^\top \tilde{r}, w_p^\top \tilde{r}) = w_p^\top cov(\tilde{r}, \tilde{r}) w_p = w_p^\top V w_p$.

A frontier portfolio has the minimum variance among portfolios that have the same expected rate of return. Thus, w_p, the N-vector portfolio weights of the frontier portfolio p, is the solution to the following quadratic program:

$$\begin{aligned} min_w \quad & \frac{1}{2} w^\top V w \\ s.t. \quad & w^\top e = E[\tilde{r}_p] \\ & w^\top \mathbb{I} = 1. \end{aligned}$$

where $\mathbb{I}$ is an N-vector of all ones. In the quadratic program, we minimize the portfolio variance subject to the constraints that portfolio expected rate of return is equal to $E[\tilde{r}_p]$ and that the portfolio weights sum to unity. Short sales (i.e., negative portfolio weights) are permitted. Therefore, the range of expected returns on feasible portfolio is unbounded. Its unique solution is:

$$w_p = \frac{CE[\tilde{r}_p] - A}{D} V^{-1} e + \frac{B - AE[\tilde{r}_p]}{D} V^{-1} \mathbb{I},$$

where $A = e^\top V^{-1} \mathbb{I} = \mathbb{I}^\top V^{-1} e$, $B = eV^{-1}e$, $C = \mathbb{I} V^{-1} \mathbb{I}$, and $D = BC - A^2$.

The variance of the frontier portfolio p is

$$\sigma^2(\tilde{r}_p) = \frac{C}{D}\left(E[\tilde{r}_p] - \frac{A}{C}\right)^2 + \frac{1}{C}.$$

This is a hyperbola in the space of mean and standard deviation:

$$\frac{\sigma^2(\tilde{r}_p)}{\frac{1}{C}} - \frac{\left(E[\tilde{r}_p] - \frac{A}{C}\right)^2}{\frac{D}{C^2}} = 1,$$

as shown in Figure 1. Its center is at $\left(0, \frac{A}{C}\right)$ and its asymptotics are $E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}} \sigma(\tilde{r}_p)$.

2.2. GENERAL MODEL

Let r_b and r_l be the riskless borrowing rate and the riskless lending rate of return on the riskless asset, respectively. Assume that the riskless borrowing rate is higher

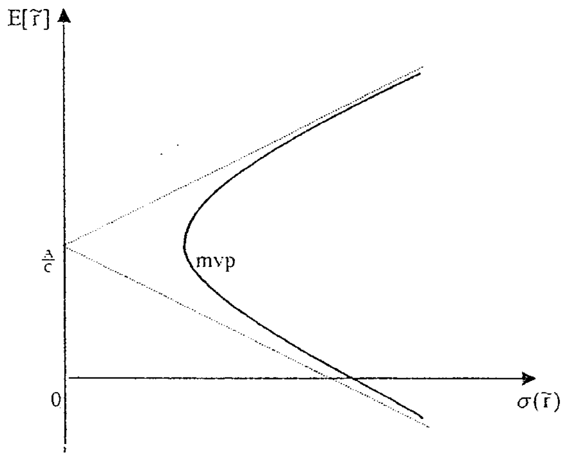

Figure 1. Portfolio Frontier in the $\sigma(\tilde{r})$-$E[\tilde{r}]$ space

than the riskless lending rate: $r_b \geqslant r_l$. Let w_p denote the N-vector portfolio weights of p on risky assets. Then $1-w_p^\top \mathbb{I}$ is the weight of riskless asset. And the rate of return on portfolio p is:

$$\tilde{r}_p = \sum_{n\in\mathcal{N}} w_n \tilde{r}_n + (1-\sum_{n\in\mathcal{N}} w_n) r = w_p^\top \tilde{r} + (1-w^\top \mathbb{I}) r(w),$$

where $r(w)$ is defined by

$$r(w) = \begin{cases} r_l, & if\ 1-w^\top \mathbb{I} \geqslant 0 \\ r_b, & if\ 1-w^\top \mathbb{I} < 0. \end{cases}$$

If $1-w^\top \mathbb{I} \geqslant 0$, the investors short sell the portfolio of N risky assets and invest (lend) the proceeds in the riskless asset, then $r(w) = r_l$. If $1-w^\top \mathbb{I} < 0$, the investors long sell the portfolio of N risky assets and short sell (borrow) the proceeds in the riskless asset, then $r(w) = r_b$.

Therefore, the expected rate of return on portfolio p is

$$E[\tilde{r}_p] = \sum_{n\in\mathcal{N}} w_n E[\tilde{r}_n] + (1-\sum_{n\in\mathcal{N}} w_n) r(w) = w_p^\top e + (1-w_p^\top \mathbb{I}) r(w).$$

It follows that $w_p^\top (e - r(w)\mathbb{I}) = E[\tilde{r}_p] - r(w)$. The variance of portofolio p is

$$\begin{aligned}\sigma^2(\tilde{r}_p) = \sigma^2(w_p^\top \tilde{r} + (1-w^\top \mathbb{I}) r(w)) &= \sigma^2(w_p^\top \tilde{r}) \\ &= w_p^\top V w_p.\end{aligned}$$

A portfolio p has the minimum variance among portfolio that have the same expected rate of return if and only if w_p, the N-vector portfolio weights of p, is the solution to the programming problem:

$$\begin{array}{ll} min_w & \frac{1}{2} w^\top V w \\ s.t. & w^\top e + (1 - w^\top \mathbb{I}) r(w) = E[\tilde{r}_p] \end{array} \tag{QP}$$

which is not a quadratic program because the function $r(w)$ is nonsmooth.

2.3. THE CASE OF THE SAME INTEREST RATE

If the riskless borrowing rate is equal to the riskless lending rate, $r(w) = r_b = r_l \equiv r$, the unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is

$$w_p = \frac{E[\tilde{r}_p] - r}{H} V^{-1}(e - r\mathbb{I})$$

where $H = (e - r\mathbb{I})^\top V^{-1}(e - r\mathbb{I}) = B - 2Ar + Cr^2 > 0$. It follows that the variance of the rate of return on portfolio p is

$$\sigma^2(\tilde{r}_p) = \frac{(E[\tilde{r}_p] - r)^2}{H}.$$

Therefore,

$$\sigma(\tilde{r}_p) = \begin{cases} -\frac{E[\tilde{r}_p] - r}{\sqrt{H}}, & if\ E[\tilde{r}_p] \leqslant r \\ \frac{E[\tilde{r}_p] - r}{\sqrt{H}}, & if\ r \leqslant E[\tilde{r}_p]. \end{cases}$$

The portfolio frontier of all assets is composed of two half-lines emanating from the point $(0, r)$ in the $\sigma(\tilde{r}_p)$-$E[\tilde{r}_p]$ plane with slopes $\sqrt{H}$ and $-\sqrt{H}$, respectly.

(0.1) $r > \frac{A}{C}$. This case is presented graphically in Figure 2, where e' is the tangent point of the half line $r - \sqrt{H}\sigma(\tilde{r}_p)$ and the portfolio frontier of all risky assets where $E[\tilde{r}_{e'}] = \frac{Ar - B}{Cr - A} = \frac{A}{C} - \frac{\frac{D}{C^2}}{r - \frac{A}{C}}$. Any portfolio on the half-line $r - \sqrt{H}\sigma(\tilde{r}_p)$ involves a long position in portfolio e'. Any portfolio on the line $r + \sqrt{H}\sigma(\tilde{r}_p)$ involves short-selling portfolio e' and investing the proceeds in the riskless asset.

(0.2) $r = \frac{A}{C}$

$$H = B - 2Ar + Cr^2 = B - 2A\frac{A}{C} + C\frac{A^2}{C^2} = \frac{D}{C} > 0$$

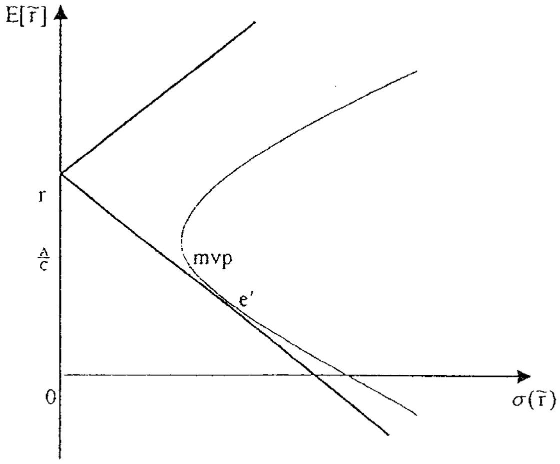

Figure 2. Portfolio frontier when $r > \dfrac{A}{C}$

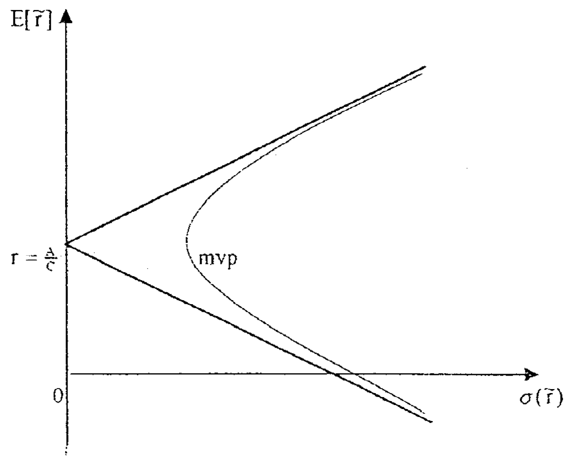

Figure 3. Portfolio Frontier when $r = \dfrac{A}{C}$

and

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$$

are the asymptotics of the portfolio frontier of risky assets which is the portfolio frontier of all assets graphed in Figure 3. Any portfolio on the portfolio frontier of all assets involves investing everything in the riskless asset and holding an arbitrage portfolio of risky assets — a portfolio whose weights sum to zero.

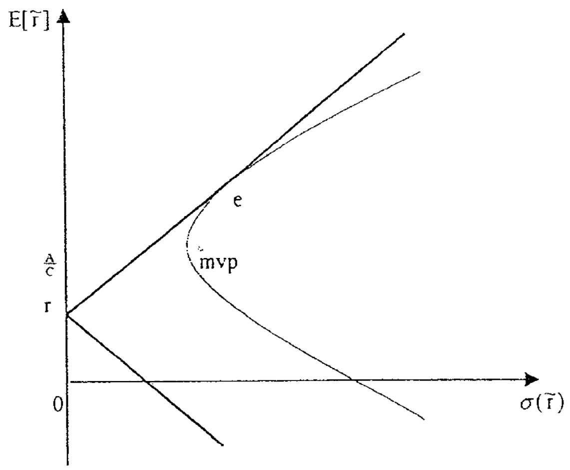

Figure 4. Portfolio Frontier when $r < \frac{A}{C}$

(0.3) $r < \frac{A}{C}$. This case is presented graphically in Figure 4, where e is the tangent point of the half line $r + \sqrt{H}\sigma(\tilde{r}_p)$ and the portfolio frontier of all risky assets where

$$E[\tilde{r}_e] = \frac{Ar - B}{Cr - A} = \frac{A}{C} - \frac{\frac{D}{C^2}}{r - \frac{A}{C}}.$$

Any portfolio on the half line $r - \sqrt{H}\sigma(\tilde{r}_p)$ involves short-selling portfolio e and investing the proceeds in the riskless asset. Any portfolio on the half line $r + \sqrt{H}\sigma(\tilde{r}_p)$ other than those on the line segment $\overline{re}$ involves short-selling the riskless asset and investing the proceeds in portfolio e.

3. Portfolio Frontier with Different Interest Rates

When the riskless borrowing rate is strictly higher than the riskless lending rate, that is, $r_b > r_l$, the function $r(w)$ in the programming problem (QP) is no longer a constant. When $w^\top \mathbb{I} \leqslant 1$, $r(w) = r_l$; and when $w^\top \mathbb{I} > 1$, $r(w) = r_b$. Then programming problem (QP) can be solved by the following two quadratic programs:

$$\begin{aligned} min_w \quad & \frac{1}{2} w^\top V w \\ s.t. \quad & w^\top e + (1 - w^\top \mathbb{I}) r_l = E[\tilde{r}_p] \\ & 1 - w^\top \mathbb{I} \geqslant 0 \end{aligned} \qquad \text{(QP1)}$$

and

$$\begin{array}{ll} min_w & \frac{1}{2} w^\top V w \\ s.t. & w^\top e + (1 - w^\top \mathbb{I}) r_b = E[\tilde{r}_p] \\ & w^\top \mathbb{I} > 1. \end{array} \tag{QP2}$$

Of the solutions to the two quadratic programs, the one with smaller variance is the solution to the programming problem (QP). We discuss (QP 1) and (QP 2) separately.

3.1. SOLUTION FOR (QP 1)

For the quadratic program (QP 1), $\nabla\{w^\top e + (1 - w^\top \mathbb{I}) r_l - E[\tilde{r}_p]\} = e - r_l \mathbb{I}$ and $\nabla\{1 - w^\top \mathbb{I}\} = -\mathbb{I}$, hence rank $\{e - r_l \mathbb{I}, -\mathbb{I}\} = 2$. Thus, any w in the constraint of the quadratic program (QP 1) is a regular point and we can apply the Kuhn-Tucker condition to find the unique solution for (QP 1).

Forming the Lagrangian, w_p is the solution to the following

$$min_{w,\lambda,\mu}\ L = \frac{1}{2} w^\top V w - \lambda\{w^\top e + (1 - w^\top \mathbb{I}) r_l - E[\tilde{r}_p]\} - \mu(1 - w^\top \mathbb{I})$$

where $\mu \geqslant 0$. The Kuhn-Tucker condition is as follows.

$$\frac{\partial L}{\partial w} = V w_p - \lambda(e - r_l \mathbb{I}) + \mu \mathbb{I} = 0, \tag{1}$$

$$\frac{\partial L}{\partial \lambda} = -\{w_p^\top e + (1 - w_p^\top \mathbb{I}) r_l - E[\tilde{r}_p]\} = 0, \tag{2}$$

$$1 - w_p^\top \mathbb{I} \geqslant 0, \tag{3}$$

$$\mu \geqslant 0, \tag{4}$$

$$\mu(1 - w_p^\top \mathbb{I}) = 0. \tag{5}$$

Solving Equation 1 for w_p gives

$$w_p = V^{-1}[\lambda(e - r_l \mathbb{I}) - \mu \mathbb{I}]. \tag{6}$$

That is, $w_p^\top = [\lambda(e - r_l \mathbb{I})^\top - \mu \mathbb{I}^\top] V^{-1}$. Right-multiplying both sides by $e - r_l \mathbb{I}$ and combine it with Equation 2, we have

$$\lambda(e - r_l \mathbb{I})^\top V^{-1}(e - r_l \mathbb{I}) - \mu \mathbb{I}^\top V^{-1}(e - r_l \mathbb{I}) = E[\tilde{r}_p] - r_l.$$

Let $H_l = (e - r_l \mathbb{I})^\top V^{-1}(e - r_l \mathbb{I}) = B - 2A r_l + C r_l^2 > 0$. Then

$$\lambda H_l - \mu(A - r_l C) = E[\tilde{r}_p] - r_l. \tag{7}$$

We consider three cases $A - r_l C$ (<0, $=0$, >0) according to $\mu = 0$ and $\mu > 0$.

First, when $\mu = 0$, Inequality 4 and Equation 5 are satisfied. From Equation 7, $\lambda = \frac{E[\tilde{r}_p] - r_l}{H_l}$. Substituting for λ and $\mu = 0$ into Equation 6, we obtain the unique solution of portfolio weights w_p for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$:

$$w_p = \frac{E[\tilde{r}_p] - r_l}{H_l} V^{-1}(e - r_l \mathbb{I}). \tag{8}$$

Hence, the variance of the rate of return on portfolio p is

$$\sigma^2(\tilde{r}_p) = w_p^\top V w_p = \frac{(E[\tilde{r}_p] - r_l)^2}{H_l}. \tag{9}$$

Note that $w_p^\top \mathbb{I} = \frac{E[\tilde{r}_p] - r_l}{H_l}(e - r_l \mathbb{I})^\top V^{-1} \mathbb{I} = \frac{E[\tilde{r}_p] - r_l}{H_l}(A - r_l C)$. By Inequality 3, we have $(E[\tilde{r}_p] - r_l)(A - r_l C) \leqslant H_l$. Therefore,

$$E[\tilde{r}_p](A - r_l C) \leqslant B - r_l A. \tag{10}$$

(1.a1) If $A - r_l C < 0$, then $r_l > \frac{A}{C}$ *and* $E[\tilde{r}_p] \geqslant \frac{B - r_l A}{A - r_l C}$.

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & if\ \dfrac{B - r_l A}{A - r_l C} \leqslant E[\tilde{r}_p] \leqslant r_l \\ \dfrac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & if\ r_l \leqslant E[\tilde{r}_p] \end{cases}$$

Then the portfolio frontier of all assets is composed of a closed line segment and an half-line in the $\sigma(\tilde{r}_p) - E[\tilde{r}_p]$ plane: the closed line segment between the point e'_l and the point $(0, r_l)$ involving the proceeding in the risky assets and the riskless asset without borrowing and lending; and the half-line emanating from the point $(0, r_l)$ with slope $\sqrt{H_l}$ involving short-selling portfolio e'_l and investing (lending) the proceeds in the riskless asset (Figure 5).

(1.b1) If $A - r_l C = 0$, then $r_l = \frac{A}{C}$ and the relation (1.10) holds for any $E[\tilde{r}_p]$.

$$H_l = B - 2Ar_l + Cr_l^2 = B - 2A\frac{A}{C} + C\frac{A^2}{C^2} = \frac{D}{C} > 0.$$

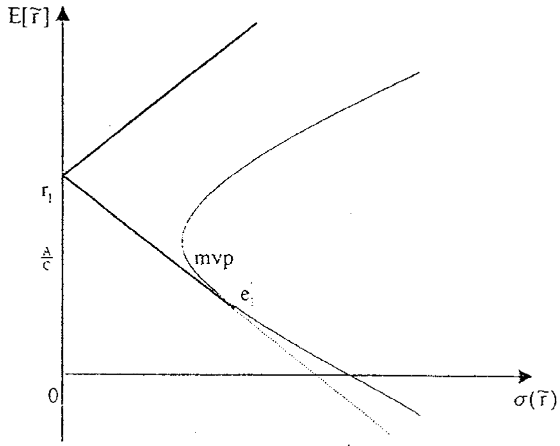

Figure 5. Portfolio frontier when $r_l > \dfrac{A}{C}$

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - \dfrac{A}{C}}{\sqrt{\dfrac{D}{C}}}, & if\ E[\tilde{r}_p] \leqslant \dfrac{A}{C} \\ \dfrac{E[\tilde{r}_p] - \dfrac{A}{C}}{\sqrt{\dfrac{D}{C}}}, & if\ \dfrac{A}{C} \leqslant E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is the asymptote of the portfolio frontier of risky assets graphed in Figure 6. Any portfolio on the portfolio frontier of all assets involves investing everything in the riskless asset and holding an arbitrage portfolio of risky assets — a portfolio with total weight sum equal to zero.

(1.c1) If $A - r_l C > 0$, then $r_l < \dfrac{A}{C}$ and $E[\tilde{r}_p] \leqslant \dfrac{B - r_l A}{A - r_l C}$.

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & if\ E[\tilde{r}_p] \leqslant r_l \\ \dfrac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & if\ r_l \leqslant E[\tilde{r}_p] \leqslant \dfrac{B - r_l A}{A - r_l C} \end{cases}$$

The portfolio frontier of all assets is composed of an half-line and a closed line segment in the $\sigma(\tilde{r}_p)$-$E[\tilde{r}_p]$ plane: the half-line emanating from the point $(0, r_l)$ with slope $-\sqrt{H_l}$ involving short-selling portfolio e_l and investing the proceeds in the riskless asset; and the closed line segment between the point $(0, r_l)$ and the

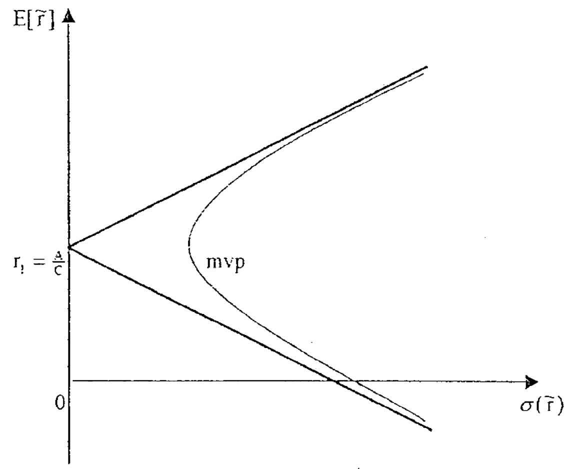

Figure 6. Portfolio frontier when $r_l = \dfrac{A}{C}$

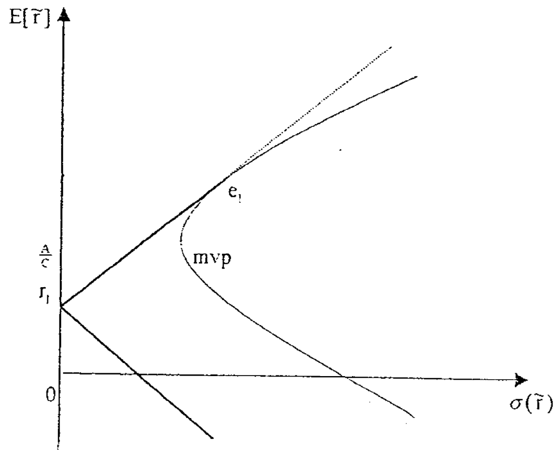

Figure 7. Portfolio Frontier when $r_l < \dfrac{A}{C}$

point e_l involving the proceeding in the risky assets and the riskless asset without borrowing and lending (Figure 7).

Secondly, when $\mu > 0$, Inequality 4 is satisfied. From Equation 5, $w_p^\top \mathbb{I} = 1$. Inequality 3 and Equation 5 are satisfied. Hence

$$\lambda(e - r_l\mathbb{I})^\top V^{-1}\mathbb{I} - \mu\mathbb{I}^\top V^{-1}\mathbb{I} = 1.$$

By Equation 6, we have

$$\lambda(A - r_l C) - \mu C = 1. \tag{11}$$

Solving λ and μ from the system of Equations 7 and 11, we have

$$\begin{cases} \lambda = \dfrac{(E[\tilde{r}_p]-r_l)C-(A-r_lC)}{D} = \dfrac{E[\tilde{r}_p]C-A}{D} \\ \mu = \dfrac{(E[\tilde{r}_p]-r_l)(A-r_lC)-H_l}{D} = \dfrac{E[\tilde{r}_p](A-r_lC)-(B-r_lA)}{D} \end{cases}$$

Substituting for λ and μ into Equation 6 gives the unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$:

$$w_p = \frac{1}{D}V^{-1}[(E[\tilde{r}_p]C-A)e+(B-E[\tilde{r}_p]A)\mathbb{I})]. \tag{12}$$

Therefore, the variance of the rate of return on portfolio p is

$$\sigma^2(\tilde{r}_p) = w_p^\top V w_p = \frac{C}{D}\left(E[\tilde{r}_p]-\frac{A}{C}\right)^2+\frac{1}{C}. \tag{13}$$

Equivalently, we can write

$$\sigma(\tilde{r}_p) = \sqrt{\frac{C}{D}\left(E[\tilde{r}_p]-\frac{A}{C}\right)^2+\frac{1}{C}}.$$

Note that $\mu > 0$ is the same as $E[\tilde{r}_p](A-r_lC)-(B-r_lA) > 0$. That is,

$$E[\tilde{r}_p](A-r_lC) > B-r_lA. \tag{14}$$

(1.a2) If $A-r_lC < 0$, then $r_l > \frac{A}{C}$ and $E[\tilde{r}_p] < \dfrac{B-r_lA}{A-r_lC}$. The portfolio frontier of all assets is composed of a part below the point e_l' of the right-branch curve of the hyperbola, in the space of standard deviation and expected rate of return, with center $\left(0, \dfrac{A}{C}\right)$ and asymptotics

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$$

involving short-selling the riskless asset and investing the proceeding in portfolio of risky assets. The portfolio frontier in the space of mean and standard deviation is presented in Figure 8.

(1.b2) If $A-r_lC = 0$, then $r_l = \dfrac{A}{C}$ and Inequality 14 does not hold.

(1.c2) If $A-r_lC > 0$, then $r_l < \dfrac{A}{C}$ and

$$E[\tilde{r}_p] > \frac{B-r_lA}{A-r_lC}.$$

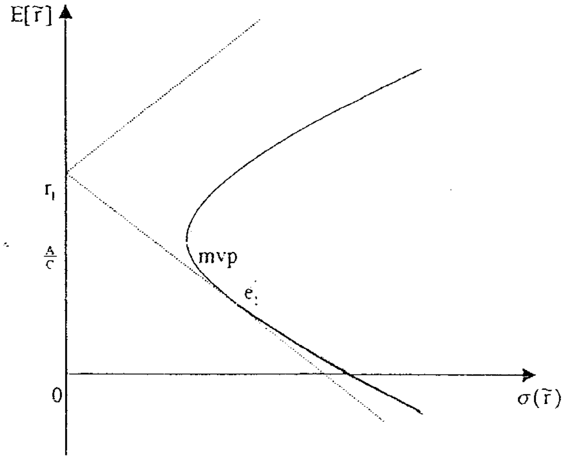

Figure 8. Portfolio frontier when $r_l > \dfrac{A}{C}$

The portfolio frontier of all assets is composed of a part above the point e_l of the right-branch curve of the hyperbola, in the space of standard deviation and expected rate of return, with center $\left(0, \dfrac{A}{C}\right)$ and asymptotics

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$$

involving short-selling (lending) the riskless asset and investing the proceeding in portfolio of risky assets. The portfolio frontier in the space of mean and standard deviation is presented in Figure 9.

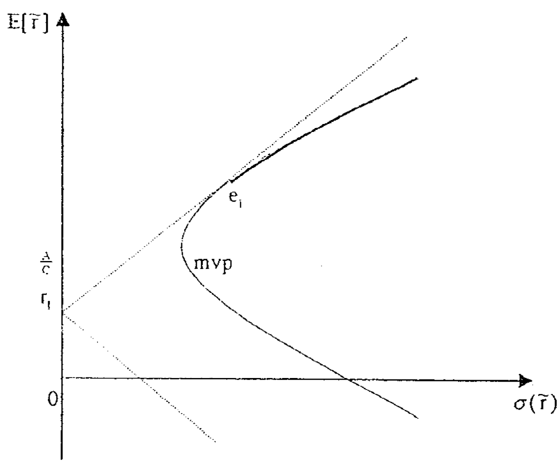

Figure 9. Portfolio frontier when $r_l < \dfrac{A}{C}$

We now summarize the above discussion for the cases $\mu=0$ and $\mu>0$.

(1.a) If $A-r_lC<0$, then $r_l>\frac{A}{C}$. The unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p=\begin{cases}\dfrac{1}{D}V^{-1}[(E[\tilde{r}_p]C-A)e+(B-E[\tilde{r}_p]A)\mathbb{I})], & if\ E[\tilde{r}_p]<\dfrac{B-r_lA}{A-r_lC}\\ \dfrac{E[\tilde{r}_p]-r_l}{H_l}V^{-1}(e-r_l\mathbb{I}), & if\ \dfrac{B-r_lA}{A-r_lC}\leqslant E[\tilde{r}_p]\end{cases}$$

and the standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p)=\begin{cases}\sqrt{\dfrac{C}{D}\left(E[\tilde{r}_p]-\dfrac{A}{C}\right)^2+\dfrac{1}{C}}, & if\ E[\tilde{r}_p]<\dfrac{B-r_lA}{A-r_lC}\\ -\dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & if\ \dfrac{B-r_lA}{A-r_lC}\leqslant E[\tilde{r}_p]\leqslant r_l\\ \dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & if\ r_l\leqslant E[\tilde{r}_p]\end{cases}$$

The portfolio frontier of all assets is composed of a curve, a closed line segment and an half-line in the $\sigma(\tilde{r}_p)-E[\tilde{r}_p]$ plane: the curve part below the point e_l' of the right-branch curve of the hyperbola with center $\left(0,\dfrac{A}{C}\right)$ and asymptotics $E[\tilde{r}_p]=\dfrac{A}{C}\pm\sqrt{\dfrac{D}{C}}\sigma(\tilde{r}_p)$ involving short-selling the riskless asset and investing the proceeding in portfolio of risky assets; the closed line segment between the point e_l' and the point $(0,r_l)$ involving the proceeding in the risky assets and the riskless asset without borrowing and lending; and the half-line emanating from the point $(0,r_l)$ with slope $\sqrt{H_l}$ involving short-selling portfolio e_l' and investing (lending) the proceeds in the riskless asset (Figure 10).

(1.b) If $A-r_lC=0$, then $r_l=\frac{A}{C}$. The unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p=\frac{E[\tilde{r}_p]-\dfrac{A}{C}}{\dfrac{D}{C}}V^{-1}\left(e-\frac{A}{C}\mathbb{I}\right),$$

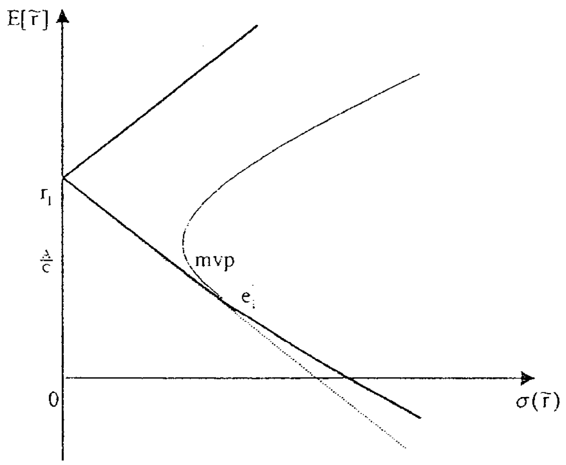

Figure 10. Portfolio frontier when $r_l > \frac{A}{C}$

and the standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - \dfrac{A}{C}}{\sqrt{\dfrac{D}{C}}}, & \text{if } E[\tilde{r}_p] \leqslant \dfrac{A}{C} \\ \dfrac{E[\tilde{r}_p] - \dfrac{A}{C}}{\sqrt{\dfrac{D}{C}}}, & \text{if } \dfrac{A}{C} \leqslant E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is the asymptotics of the portfolio frontier of risky assets. Any portfolio on the portfolio frontier of all assets involves investing everything in the riskless asset and holding an arbitrage portfolio of risky assets — a portfolio whose weights sum to zero (Figure 11).

(1.c) If $A - r_l C > 0$, then $r_l < \frac{A}{C}$. The unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p = \begin{cases} \dfrac{E[\tilde{r}_p] - r_l}{H_l} V^{-1}(e - r_l \mathbb{I}), & \text{if } E[\tilde{r}_p] \leqslant \dfrac{B - r_l A}{A - r_l C} \\ \dfrac{1}{D} V^{-1}[(E[\tilde{r}_p]C - A)e + (B - E[\tilde{r}_p]A)\mathbb{I})], & \text{if } \dfrac{B - r_l A}{A - r_l C} < E[\tilde{r}_p] \end{cases}$$

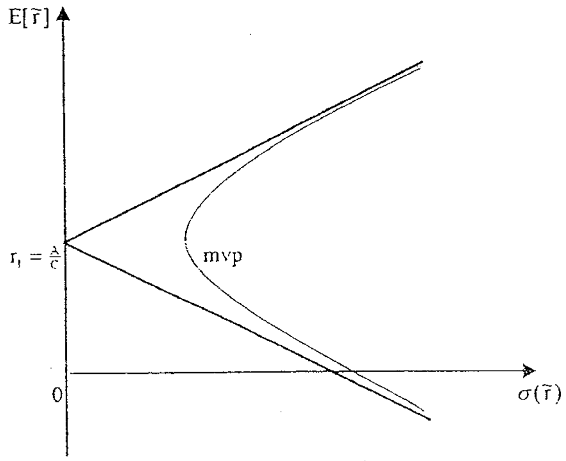

Figure 11. Portfolio Frontier when $r_l = \frac{A}{C}$

and the standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & \mathit{if}\ E[\tilde{r}_p] \leqslant r_l \\ \dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & \mathit{if}\ r_l \leqslant E[\tilde{r}_p] \leqslant \dfrac{B-r_lA}{A-r_lC} \\ \sqrt{\dfrac{C}{D}\left(E[\tilde{r}_p]-\dfrac{A}{C}\right)^2+\dfrac{1}{C}}, & \mathit{if}\ \dfrac{B-r_lA}{A-r_lC} < E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is composed of an half-line, a closed line segment and a curve in the $\sigma(\tilde{r}_p)-E[\tilde{r}_p]$ plane: the half-line emanating from the point $(0, r_l)$ with slope $-\sqrt{H_l}$ involving short-selling portfolio e_l and investing the proceeds in the riskless asset; the closed line segment between the point $(0, r_l)$ and the point e_l involving the proceeding in the risky assets and the riskless asset without borrowing and lending; and the curve part above the point e_l of the right-branch curve of the hyperbola with center $\left(0, \frac{A}{C}\right)$ and asymptotics $E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$ involving short-selling (lending) the riskless asset and investing the proceeding in portfolio of risky assets (Figure 12).

3.2. SOLUTION FOR (QP 2)

We now solve the quadratic program (QP 2). Similarly, any w in the constraint of the quadratic program (QP 2) is a regular point and we can use the Kuhn–Tucker condition for the unique solution to the quadratic program (QP 2).

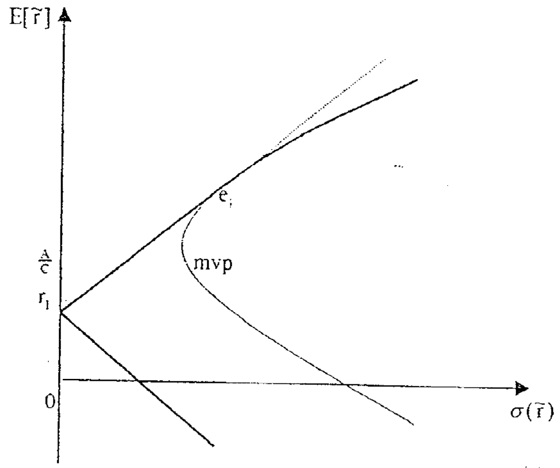

Figure 12. Portfolio Frontier when $r_l < \dfrac{A}{C}$

Forming the Lagrangian, w_p is the solution to the following

$$min_{w,\lambda,\mu} L = \frac{1}{2} w^\top V w - \lambda\{w^\top e + (1 - w^\top \mathbb{I}) r_b - E[\tilde{r}_p]\} - \mu\{w^\top \mathbb{I} - 1\}$$

where μ is a positive constant. Then Kuhn–Tucker condition is as follow.

$$\frac{\partial L}{\partial w} = V w_p - \lambda(e - r_b \mathbb{I}) - \mu \mathbb{I} = 0, \tag{15}$$

$$\frac{\partial L}{\partial \lambda} = -\{w_p^\top e + (1 - w_p^\top \mathbb{I}) r_b - E[\tilde{r}_p]\} = 0, \tag{16}$$

$$w_p^\top \mathbb{I} > 1, \tag{17}$$

$$\mu \geqslant 0, \tag{18}$$

$$\mu(w_p^\top \mathbb{I} - 1) = 0. \tag{19}$$

From Inequality 17 and Equation 19, we have $\mu = 0$. Solving Equation 15 for w_p gives

$$w_p = \lambda V^{-1}(e - r_b \mathbb{I}) \tag{20}$$

which gives $w_p^\top = \lambda(e - r_b \mathbb{I})^\top V^{-1}$. Right-multiplying both sides by $e - r_b \mathbb{I}$ and applying Equation 16, we have

$$\lambda H_b = E[\tilde{r}_p] - r_b$$

where $H_b = (e - r_b \mathbb{I})^\top V^{-1}(e - r_b \mathbb{I}) = B - 2A r_b + C r_b^2 > 0$. Solving it for λ, we have

$$\lambda = \frac{E[\tilde{r}_p] - r_b}{H_b}. \tag{21}$$

Combining Equation 20 and Equation 21 gives the unique set of portfolio weights for the portfolio p to the expected rate of return of $E[\tilde{r}_p]$ and the variance of portfolio p:

$$w_p = \frac{E[\tilde{r}_p] - r_b}{H_b} V^{-1}(e - r_b \mathbb{I}), \tag{22}$$

$$\sigma^2(\tilde{r}_p) = w_p^\top V w_p = \frac{(E[\tilde{r}_p] - r_b)^2}{H_b}. \tag{23}$$

Equivalently, we can write

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}, & if\ E[\tilde{r}_p] \leqslant r_b \\ \dfrac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}, & if\ r_b \leqslant E[\tilde{r}_p]. \end{cases}$$

Note that $w_p^\top \mathbb{I} = \dfrac{E[\tilde{r}_p] - r_b}{H_b}(e - r_b\mathbb{I})^\top V^{-1}\mathbb{I} = \dfrac{E[\tilde{r}_p] - r_b}{H_b}(A - r_b C)$. Equation 17 becomes $(E[\tilde{r}_p] - r_b)(A - r_b C) > H_b$. It follows that

$$E[\tilde{r}_p](A - r_b C) > B - r_b A. \tag{24}$$

(2.a) If $A - r_b C < 0$, then $r_b > \dfrac{A}{C}$ and $E[\tilde{r}_p] < \dfrac{B - r_b A}{A - r_b C} \leqslant r_b$. Therefore

$$\sigma(\tilde{r}_p) = -\frac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}.$$

The portfolio frontier of all assets is composed of an half-line emanating from the point e_b' in the $\sigma(\tilde{r}_p)$-$E[\tilde{r}_p]$ plane with slope $-\sqrt{H_b}$. This involves in short-selling of the riskless asset and investing the proceeding in the portfolio e_b' (Figure 13).

(2.b) If $A - r_b C = 0$, then $r_b = \dfrac{A}{C}$ and the relation (2.12) does not hold.

(2.c) If $A - r_b C > 0$, then $r_b < \dfrac{A}{C}$ and $E[\tilde{r}_p] > \dfrac{B - r_b A}{A - r_b C} \geqslant r_b$. Therefore,

$$\sigma(\tilde{r}_p) = \frac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}.$$

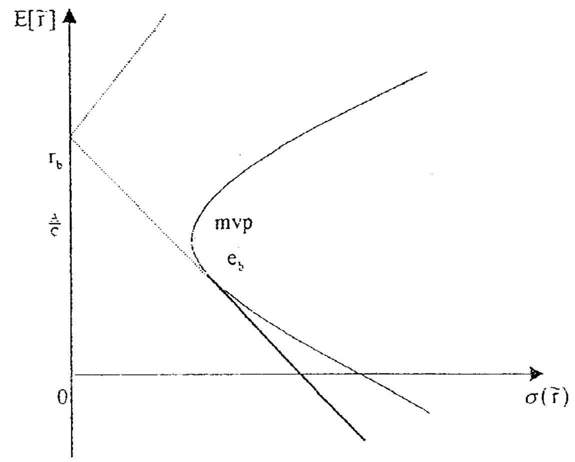

Figure 13. Portfolio Frontier when $r_b > \dfrac{A}{C}$

The portfolio frontier of all assets is composed of an half-line emanating from the point e_l in the $\sigma(\tilde{r}_p) - E[\tilde{r}_p]$ plane with slope $\sqrt{H_b}$. It again involves in short-selling (borrowing) the riskless asset and investing the proceeding in the portfolio e_b (Figure 14).

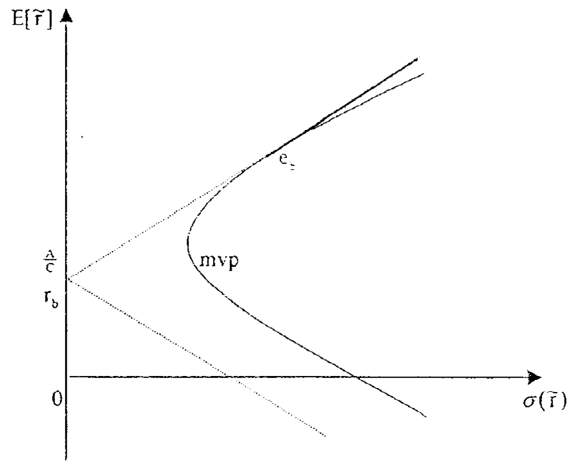

Figure 14. Portfolio Frontier when $r_b < \dfrac{A}{C}$

3.3. SUMMARY

Here we summarize our results Sections 3.1 and 3.2 for the programming program (QP) where the riskless borrowing rate is higher than the riskless lending rate, $r_b > r_l$.

(a) $\frac{A}{C} < r_l < r_b$, we first solve an important point P (See Figure 15). The point P is the intersection point between two half-line, one emanating from the point $(0, r_l)$ with slope $-\sqrt{H_l}$, that is, $E[\tilde{r}_p] = r_l - \sqrt{H_l}\sigma(\tilde{r}_p)$, and another emanating from the point $(0, r_b)$ with slope $-\sqrt{H_b}$, that is, $E[\tilde{r}_p] = r_b - \sqrt{H_b}\sigma(\tilde{r}_p)$. Then $P\left(\frac{r_b - r_l}{\sqrt{H_b} - \sqrt{H_l}}, \frac{\sqrt{H_b}r_l - \sqrt{H_l}r_b}{\sqrt{H_b} - \sqrt{H_l}}\right)$.

The unique set of portfolio weights for the optimal portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p = \begin{cases} \frac{E[\tilde{r}_p] - r_b}{H_b} V^{-1}(e - r_b\mathbb{I}), & \textit{if } E[\tilde{r}_p] \leqslant E[\tilde{r}_P] \\ \frac{E[\tilde{r}_p] - r_l}{H_l} V^{-1}(e - r_l\mathbb{I}), & \textit{if } E[\tilde{r}_P] \leqslant E[\tilde{r}_p] \end{cases}$$

The standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\frac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}, & \textit{if } E[\tilde{r}_p] \leqslant E[\tilde{r}_P] \\ -\frac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & \textit{if } E[\tilde{r}_P] \leqslant E[\tilde{r}_p] \leqslant r_l \\ \frac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & \textit{if } r_l \leqslant E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is composed of two half-lines and a closed line segment in the $\sigma(\tilde{r}_p) - E[\tilde{r}_p]$ plane: the half-line emanating from the point P with slope $-\sqrt{H_b}$ involving short-selling the riskless asset and investing the the portfolio e_b'; the closed line segment between the point $(0, r_l)$ and the point P involving investing the proceeding in the risky assets and the riskless asset without

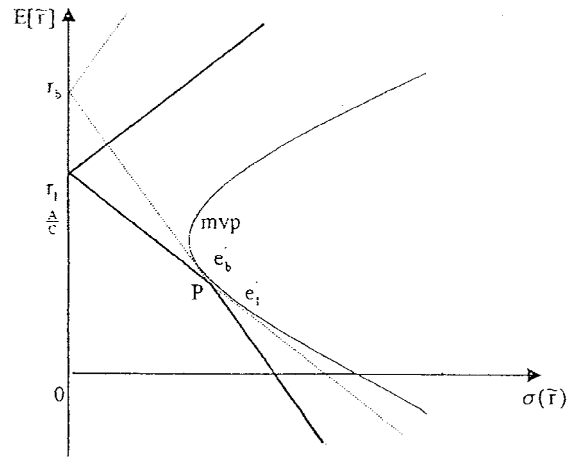

Figure 15. Portfolio Frontier when $\frac{A}{C} < r_l < r_b$

borrowing and lending; and the half-line emanating from the point $(0, r_l)$ with slope $\sqrt{H_l}$ involving short-selling portfolio e_l' and investing (lending) the proceeds in the riskless asset.

(b) $\frac{A}{C} = r_l < r_b$, we first solve for the point P (See Figure 16), the intersection point between two half-line: one emanating from the point $\left(0, \frac{A}{C}\right)$ with slope $-\sqrt{\frac{D}{C}}$, that is, $E[\tilde{r}_p] = \frac{A}{C} - \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p)$, and another emanating from the point $(0, r_b)$ with slope $-\sqrt{H_b}$, that is, $E[\tilde{r}_p] = r_b - \sqrt{H_b}\sigma(\tilde{r}_p)$. Then

$$P\left(\frac{r_b - \frac{A}{C}}{\sqrt{H_b} - \sqrt{\frac{D}{C}}}, \frac{\sqrt{H_b}\frac{A}{C} - \sqrt{\frac{D}{C}} r_b}{\sqrt{H_b} - \sqrt{\frac{D}{C}}}\right).$$

The unique set of portfolio weights for the optimal portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p = \begin{cases} \dfrac{E[\tilde{r}_p] - r_b}{H_b} V^{-1}(e - r_b\mathbb{I}), & \text{if } E[\tilde{r}_p] \leqslant E[\tilde{r}_P] \\ \dfrac{E[\tilde{r}_p] - \frac{A}{C}}{\frac{D}{C}} V^{-1}\left(e - \frac{A}{C}\mathbb{I}\right), & \text{if } E[\tilde{r}_P] \leqslant E[\tilde{r}_p] \end{cases}$$

The standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p] - r_b}{\sqrt{H_b}}, & \text{if } E[\tilde{r}_p] \leqslant E[\tilde{r}_P] \\ -\dfrac{E[\tilde{r}_p] - \frac{A}{C}}{\sqrt{\frac{D}{C}}}, & \text{if } E[\tilde{r}_P] \leqslant E[\tilde{r}_p] \leqslant \frac{A}{C} \\ \dfrac{E[\tilde{r}_p] - \frac{A}{C}}{\sqrt{\frac{D}{C}}}, & \text{if } \frac{A}{C} \leqslant E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is composed of two half-lines and a closed line segment in the $\sigma(\tilde{r}_p) - E[\tilde{r}_p]$ plane: the half-line emanating from the point P with slope $-\sqrt{H_b}$ involving short-selling the riskless asset and investing the the portfolio e_b'; the closed line segment between the point $\left(0, \frac{A}{C}\right)$ and the point P and the half-line emanating from the point $\left(0, \frac{A}{C}\right)$ with slope $\sqrt{\frac{D}{C}}$ involving investing everything in the riskless asset and holding an arbitrage portfolio of risky assets — a portfolio whose weights sum to zero.

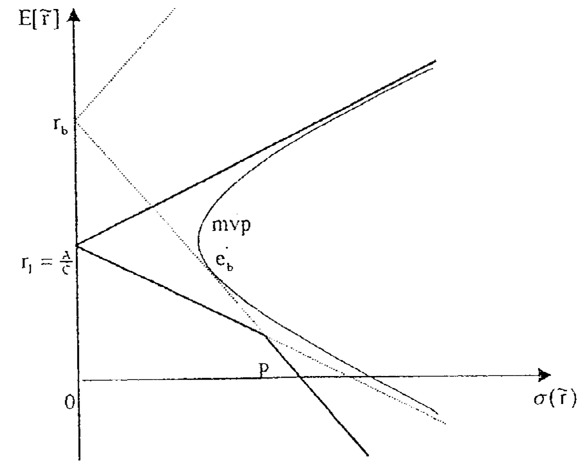

Figure 16. Portfolio Frontier when $\frac{A}{C}=r_l<r_b$

(c) $r_l<\frac{A}{C}<r_b$. We discuss the two settings: $r_b-\frac{A}{C}>\frac{A}{C}-r_l$ and $r_b-\frac{A}{C}\leqslant\frac{A}{C}-r_l$.

(c.1) When $r_b-\frac{A}{C}>\frac{A}{C}-r_l$, $H_b>H_l$. We first find the point P (See Figure 17), the intersection point between two half-line: one emanating from the point $(0,r_l)$ with slope $-\sqrt{H_l}$, that is, $E[\tilde{r}_p]=r_l-\sqrt{H_l}\sigma(\tilde{r}_p)$, and another emanating from the point $(0,r_b)$ with slope $-\sqrt{H_b}$, that is, $E[\tilde{r}_p]=r_b-\sqrt{H_b}\sigma(\tilde{r}_p)$. Then $P\left(\frac{r_b-r_l}{\sqrt{H_b}-\sqrt{H_l}},\frac{\sqrt{H_b}r_l-\sqrt{H_l}r_b}{\sqrt{H_b}-\sqrt{H_l}}\right)$.

The unique set of portfolio weights for the optimal portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p=\begin{cases}\dfrac{E[\tilde{r}_p]-r_b}{H_b}V^{-1}(e-r_b\mathbb{I}), & if\ E[\tilde{r}_p]\leqslant E[\tilde{r}_P]\\ \dfrac{E[\tilde{r}_p]-r_l}{H_l}V^{-1}(e-r_l\mathbb{I}), & if\ E[\tilde{r}_P]\leqslant E[\tilde{r}_p]\leqslant\dfrac{B-r_lA}{A-r_lC}\\ \dfrac{1}{D}V^{-1}[(E[\tilde{r}_p]C-A)e+(B-E[\tilde{r}_p]A)\mathbb{I}], & if\ \dfrac{B-r_lA}{A-r_lC}<E[\tilde{r}_p]\end{cases}$$

The standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p)=\begin{cases} -\dfrac{E[\tilde{r}_p]-r_b}{\sqrt{H_b}}, & if\ E[\tilde{r}_p]\leqslant E[\tilde{r}_P] \\ -\dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & if\ E\tilde{r}_P\leqslant E[\tilde{r}_p]\leqslant r_l \\ \dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & if\ r_l\leqslant E[\tilde{r}_p]\leqslant \dfrac{B-r_lA}{A-r_lC} \\ \sqrt{\dfrac{C}{D}\left(E[\tilde{r}_p]-\dfrac{A}{C}\right)^2+\dfrac{1}{C}}, & if\ \dfrac{B-r_lA}{A-r_lC}<E[\tilde{r}_p] \end{cases}$$

The portfolio frontier of all assets is composed of an half-line, two closed line segments and a curve in the $\sigma(\tilde{r}_p)-E[\tilde{r}_p]$ plane: the half-line emanating from the point P with slope $-\sqrt{H_b}$ involving short-selling the riskless asset and investing the the portfolio e_b'; the closed line segment between the point $(0,r_l)$ and the point P involving short-selling portfolio e_l and investing the proceeds in the riskless asset; the closed line segment between the point $(0,r_l)$ and the point e_l involving investing the proceeding in the risky assets and the riskless asset without borrowing and lending; and the curve part above the point e_l of the right-branch curve of the hyperbola with center $\left(0,\dfrac{A}{C}\right)$ and asymptotics $E[\tilde{r}_p]=\dfrac{A}{C}\pm\sqrt{\dfrac{D}{C}}\sigma(\tilde{r}_p)$ involving short-selling (lending) the riskless asset and investing the proceeding in the portfolio of the risky assets.

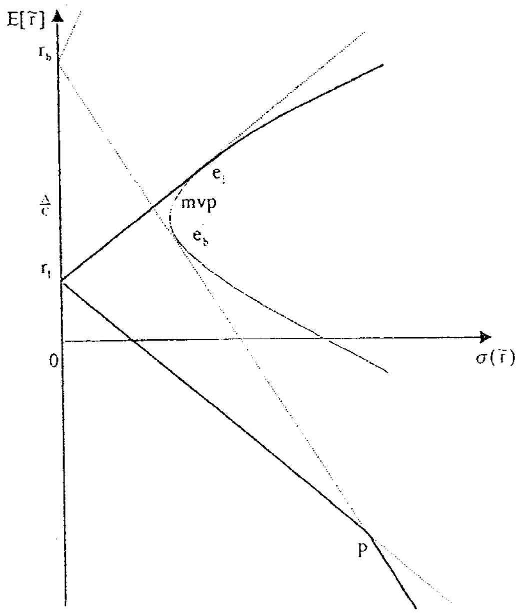

Figure 17. Portfolio Frontier when $r_l<\dfrac{A}{C}<r_b$ and $r_b-\dfrac{A}{C}>\dfrac{A}{C}-r_l$

(c.2) When $r_b - \frac{A}{C} > \frac{A}{C} - r_l$. The unique set of portfolio weights for the optimal portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p = \begin{cases} \frac{E[\tilde{r}_p] - r_l}{H_l} V^{-1}(e - r_l \mathbb{I}), & \text{if } E[\tilde{r}_p] \leqslant \frac{B - r_l A}{A - r_l C} \\ \frac{1}{D} V^{-1}[(E[\tilde{r}_p]C - A)e + (B - E[\tilde{r}_p]A)\mathbb{I}], & \text{if } \frac{B - r_l A}{A - r_l C} < E[\tilde{r}_p]. \end{cases}$$

The standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\frac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & \text{if } E[\tilde{r}_p] \leqslant r_l \\ \frac{E[\tilde{r}_p] - r_l}{\sqrt{H_l}}, & \text{if } r_l \leqslant E[\tilde{r}_p] \leqslant \frac{B - r_l A}{A - r_l C} \\ \sqrt{\frac{C}{D}\left(E[\tilde{r}_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}, & \text{if } \frac{B - r_l A}{A - r_l C} < E[\tilde{r}_p]. \end{cases}$$

The portfolio frontier of all assets is composed of an half-line, a closed line segment and a curve in the $\sigma(\tilde{r}_p) - E[\tilde{r}_p]$ plane: the half-line emanating from the point $(0, r_l)$ with slope $-\sqrt{H_l}$ involving short-selling portfolio e_l and investing the proceeds in the riskless asset; the closed line segment between the point $(0, r_l)$ and the point e_l involving the proceeding in the risky assets and the riskless asset without borrowing and lending; and the curve part above the point e_l of the right-branch curve of the hyperbola with center $\left(0, \frac{A}{C}\right)$ and asymptotics $E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}} \sigma(\tilde{r}_p)$ involving short-selling (lending) the riskless asset and investing the proceeding in the portfolio of the risky assets (Figure 18).

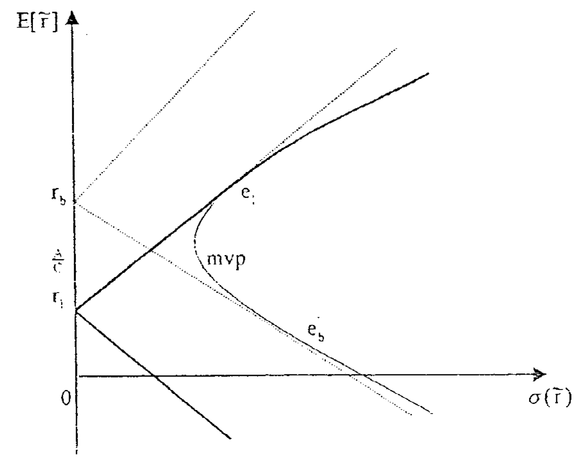

Figure 18. Portfolio Frontier when $r_l < \frac{A}{C} < r_b$ and $r_b - \frac{A}{C} \leqslant \frac{A}{C} - r_l$

(d) $r_l < r_b < \dfrac{A}{C}$. The unique set of portfolio weights for the optimal portfolio p having an expected rate of return of $E[\tilde{r}_p]$ is as follows.

$$w_p = \begin{cases} \dfrac{E[\tilde{r}_p]-r_l}{H_l} V^{-1}(e-r_l\mathbb{I}), & \text{if } E[\tilde{r}_p] \leqslant \dfrac{B-r_lA}{A-r_lC} \\ \dfrac{1}{D} V^{-1}[(E[\tilde{r}_p]C-A)e+(B-E[\tilde{r}_p]A)\mathbb{I})], & \text{if } \dfrac{B-r_lA}{A-r_lC} < E[\tilde{r}_p] \leqslant \dfrac{B-r_bA}{A-r_bC} \\ \dfrac{E[\tilde{r}_p]-r_b}{H_b} V^{-1}(e-r_b\mathbb{I}), & \text{if } \dfrac{B-r_bA}{A-r_bC} < E[\tilde{r}_p]. \end{cases}$$

The standard deviation of the rate of return on portfolio p is

$$\sigma(\tilde{r}_p) = \begin{cases} -\dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & \text{if } E[\tilde{r}_p] \leqslant r_l \\ \dfrac{E[\tilde{r}_p]-r_l}{\sqrt{H_l}}, & \text{if } r_l \leqslant E[\tilde{r}_p] \leqslant \dfrac{B-r_lA}{A-r_lC} \\ \sqrt{\dfrac{C}{D}\left(E[\tilde{r}_p]-\dfrac{A}{C}\right)^2+\dfrac{1}{C}}, & \text{if } \dfrac{B-r_lA}{A-r_lC} < E[\tilde{r}_p] \leqslant \dfrac{B-r_bA}{A-r_bC} \\ \dfrac{E[\tilde{r}_p]-r_b}{\sqrt{H_b}}, & \text{if } \dfrac{B-r_bA}{A-r_bC} < E[\tilde{r}_p]. \end{cases}$$

The portfolio frontier of all assets is composed of two half-lines, a closed line segment and a closed curve in the $\sigma(\tilde{r}_p)$-$E[\tilde{r}_p]$ plane: the half-line emanating from the point $(0, r_l)$ with slope $-\sqrt{H_l}$ involving short-selling portfolio e_l and investing the proceeds in the riskless asset; the closed line segment between the point $(0, r_l)$ and the point e_l involving short-selling the portfolio e_l and investing (lending) the proceeds in the riskless asset; the closed curve part between the point e_l and the point e_b of the right-branch curve of the hyperbola with center $\left(0, \dfrac{A}{C}\right)$ and asymptotics $E[\tilde{r}_p] = \dfrac{A}{C} \pm \sqrt{\dfrac{D}{C}}\sigma(\tilde{r}_p)$ involving the proceeding in the risky assets and the riskless asset without borrowing and lending; and the half-line emanating from the point e_b with slope $\sqrt{H_b}$ involving short-selling (borrowing) the riskless asset and investing the proceeding in the portfolio e_b (Figure 19).

4. Conclusions and Remarks

Huang and Litzenberger (1988) studied the mathematics of the portfolio frontier in Chapter 3. The portfolio frontier of all risky assets is the right-branch of hyperbola in the standard deviation–expected rate of return space while the portfolio frontier of all risky assets and a riskless asset is two half-lines in the space. In our model when riskless borrowing and lending interest rates are different, the portfolio frontier of all risky assets and a riskless asset is a continuous (smooth) curve with various simple and basic portfolio frontiers in the space. This is important for

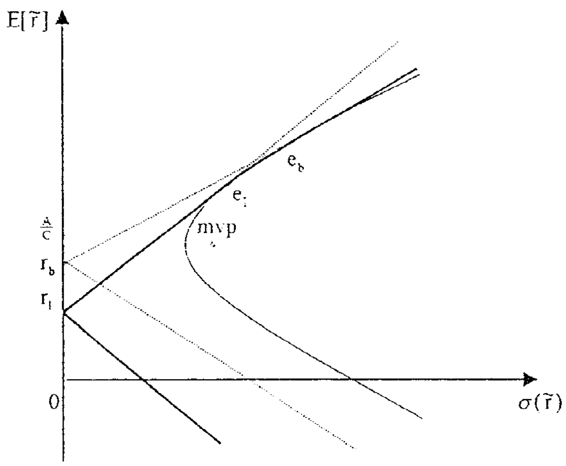

Figure 19. Portfolio Frontier when $r_l < r_b < \frac{A}{C}$

establishing various properties. For example, it follows from our analysis that there exists two (mutual) fund separation or one (mutual) fund separation. The capital asset pricing model (CAPM) also holds in this setting.

We solve the portfolio selection problem with different interest rates for borrowing and lending by the Kuhn–Tucker condition of the programming problem (QP). How do we solve the problem as follows?

$$\begin{aligned} & min_w \ w^\top V w (0) \\ & s.t. \ \ w \in D \end{aligned}$$

where V is a positive defined matrix, D is a convex set, $D = \cup_{k=1}^{2} D_k$ and D_1, D_2 are unintersection convex sets $D_1 \cap D_2 = \phi$. In fact, if w_k is the solution to the following problem

$$\begin{aligned} & min_w \quad w^\top V w \\ & s.t. \quad w \in D_k \end{aligned}$$

for $k = 1,2$ and $w_0^\top V w_0 = min\{w_k^\top V w_k, k = 1,2\}$, then w_0 is the solution to programming problem (0). Specifically, $D = D_l \cup D_b$ where

$$\begin{aligned} D_l &= \{w \in \mathcal{R}^N \mid w^\top e + (1 - w^\top \mathbb{I}) r_l = E[\tilde{r}_p] \quad and \quad w^\top \mathbb{I} \leqslant 1\} \\ D_b &= \{w \in \mathcal{R}^N \mid w^\top e + (1 - w^\top \mathbb{I}) r_b = E[\tilde{r}_p] \quad and \quad w^\top \mathbb{I} > 1\} \end{aligned}$$

Our model, as that of Markowitz (1952), studies the problem of two-period portfolio frontier. For continuous-time economy, Cvitanić and Karatzas (1992) developed a theory for the classical consumption/investment problem when the portfolio is constrained to take values in a given closed, convex and nonempty set. They adopted an Ito process model for the financial market with one bond and many stocks, and studied in its framework the stochastic control problem of maximizing expected utility from terminal wealth and/or consumption. Appendix B therein applied the convex duality methodology to the important consumption/investment problem with a high interest rate for borrowing.

In fact, the theory of option pricing developed by Black and Scholes (1973), and Merton (1973) is built on the idea that the option price should be equal to the cost of initiating a dynamic trading strategy in the primary assets of bond and stocks so as to guarantee the no-arbitrage condition. This approach is further generalized to consider economies with friction by Cvitanić and Karatzas (1993), Karoui and Quenez (1995), and Munk (1997).

5. Appendix. Solution for (QP 2)

We can solve the the quadratic program (QP 2) by using the method of artificail variable. Let $z > 0$ be the artificail variable, we rewrite the quadratic program (QP 2) as follows

$$\begin{array}{ll} min_w & \frac{1}{2} w^\top V w \\ s.t. & w^\top e + (1 - w^\top \mathbb{I}) r_b = E[\tilde{r}_p] \\ & w^\top \mathbb{I} = 1 + z. \end{array}$$

Forming the Lagrangian, w_p is the solution to the following

$$min_{w,\lambda,\mu} L = \frac{1}{2} w^\top V w - \lambda\{w^\top e + (1 - w^\top \mathbb{I}) r_b - E[\tilde{r}_p]\} - \mu\{w^\top \mathbb{I} - (1+z)\}$$

where λ and μ are constants. The first order necessary and sufficient condition for w_p to be the solution is

$$\frac{\partial L}{\partial w} = V w_p - \lambda(e - r_b \mathbb{I}) - \mu \mathbb{I} = 0, \tag{25}$$

$$\frac{\partial L}{\partial \lambda} = -\{w_p^\top e + (1 - w_p^\top \mathbb{I}) r_b - E[\tilde{r}_p]\} = 0, \tag{26}$$

$$\frac{\partial L}{\partial \mu} = -\{w_p^\top \mathbb{I} - (1+z)\} = 0, \tag{27}$$

that is,

$$\begin{aligned} V w_p &= \lambda(e - r_b \mathbb{I}) + \mu \mathbb{I} \\ w_p^\top (e - r_b \mathbb{I}) &= E[\tilde{r}_p] - r_b \\ w_p^\top \mathbb{I} &= 1 + z. \end{aligned}$$

Solving Equation 25 for w_p gives

$$w_p = \lambda V^{-1}(e - r_b \mathbb{I}) + \mu V^{-1} \mathbb{I} \tag{28}$$

that is, $w_p^\top = \lambda(e - r_b\mathbb{I})^\top V^{-1} + \mu\mathbb{I}^\top V^{-1}$. Right-multiplying both sides by $e - r_b\mathbb{I}$ and $\mathbb{I}$ gives

$$\begin{aligned}\lambda H_b + \mu(A - r_b C) &= E[\tilde{r}_p] - r_b \\ \lambda(A - r_b C) + \mu C &= 1 + z\end{aligned}$$

from Equations 26 and 27, then

$$\begin{aligned}\lambda &= \frac{(E[\tilde{r}_p] - r_b)C - (1+z)(A - r_b C)}{D} \\ \mu &= \frac{(1+z)H_b - (A - r_b C)(E[\tilde{r}_p] - r_b)}{D}\end{aligned}$$

Substituting for λ and μ into Equation 28 gives the unique set of portfolio weights for the portfolio p having an expected rate of return of $E[\tilde{r}_p]$:

$$\begin{aligned}w_p = \frac{1}{D}V^{-1}\{&[(E[\tilde{r}_p] - r_b)C - (1+z)(A - r_b C)]e \\ &+ [(1+z)(B - r_b A) - A(E[\tilde{r}_p] - r_b)]\mathbb{I}\}\end{aligned} \tag{29}$$

hence the variance of the rate of return on portfolio p is

$$\sigma^2(\tilde{r}_p) = w_p^\top V w_p = \frac{H_b}{D}\left(1 + z - \frac{(E[\tilde{r}_p] - r_b)(A - r_b C)}{H_b}\right)^2 + \frac{(E[\tilde{r}_p] - r_b)^2}{H_b} \tag{30}$$

When $\dfrac{(E[\tilde{r}_p] - r_b)(A - r_b C)}{H_b} \leqslant 1$, $z = 0$ is such that $\sigma^2(\tilde{r}_p)$ takes the minimum value, but it contradicts the assumption $z > 0$. So $\dfrac{(E[\tilde{r}_p] - r_b)(A - r_b C)}{H_b} > 1$ and

$$1 + z = \frac{(E[\tilde{r}_p] - r_b)(A - r_b C)}{H_b}$$

$\sigma^2(\tilde{r}_p)$ takes the minimum value $\dfrac{(E[\tilde{r}_p] - r_b)^2}{H_b}$ which is Equation 23, and

$$\begin{aligned}w_p &= \frac{1}{D}V^{-1}\left[(E[\tilde{r}_p] - r_b)C - \frac{(E[\tilde{r}_p] - r_b)(A - r_b C)}{H_b}(A - r_b C)\right](e - r_b\mathbb{I}) \\ &= \frac{E[\tilde{r}_p] - r_b}{H_l}V^{-1}(e - r_b\mathbb{I})\end{aligned}$$

from Equation (29), which is Equation 22.

Note $z > 0$ is $(E[\tilde{r}_p] - r_b)(A - r_b C) > H_b$, that is, $E[\tilde{r}_p](A - r_b C) > B - r_l A$ which is Equation 24.

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